

The integral representation of solutions of KZ equation and a modification by \mathcal{K} operator insertion

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Abstract

A root of unity limit of the q -deformed Virasoro algebra is considered. The $\widehat{sl}(2)_k$ current algebra and the integral formulas of the solutions of the KZ equations can be realized by the q -deformed boson at the limit and an additional boson. We explicitly construct the integral representation of the four-point blocks with a \mathcal{K} -operator insertion.

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1 Introduction

The AGT relation [1] states that the instanton partition functions [2] of the four-dimensional $\mathcal{N} = 2$ $SU(2)$ gauge theory are related to the correlation functions [3] of the two-dimensional conformal field theory with Virasoro symmetry. The extension to the similar correspondence between $SU(n)$ gauge theory and conformal field theory with W_n symmetry has been constructed in [4, 5]. Since then, the both sides of the correspondence have been intensively studied by a number of people. For example, see [9–32].

There exists a natural generalization to the connection between the 2d theory with the q -deformed Virasoro/ W symmetry and the five-dimensional gauge theory [33–39]. Recently, the elliptic Virasoro/6d correspondence is also proposed in [40, 41].

In our previous papers [42–44], we considered a r -th root of unity limit in q and t ($q \rightarrow \omega$, $t \rightarrow \omega$ with $\omega = e^{\frac{2\pi i}{r}}$) of the q -W/5d correspondence. In [42], we proposed a limiting procedure to get the Virasoro/ W block in the 2d side from that in the q -deformed version. In [43], we have elaborated the limiting procedure and showed that the \mathbf{Z}_r -parafermionic CFT appears in the 2d side.¹

The another extension of the AGT relation by including various defects in the gauge theory have also been considered [46]. We are interested in the defects, the so-called surface operator, supported on two-dimensional submanifolds (for review for surface operator, see [47]). There are two kinds of the surface operators in a sense, i.e. two-dimensional defects brought either from 2d-defect or 4d-defects in the M5-brane construction of $\mathcal{N} = 2$ gauge theory. The instanton partition functions in the presence of a kind of surface operator are related with the conformal block with a degenerate field insertion [48–54]. In the present paper, we focus on the 4d-defects. It was conjectured that the instanton partition functions in this case are related to the affine $\widehat{sl}_k(n)$ current block with a mysterious operator, the so-called \mathcal{K} operator, insertion [55, 56]. More general cases of the surface operator insertion are discussed in [57, 58].

On the other hand, the defining relation of the q -Virasoro algebra is [59–61]

$$f(z'/z)\mathcal{T}(z)\mathcal{T}(z') - f(z/z')\mathcal{T}(z')\mathcal{T}(z) = \frac{(1-q)(1-t^{-1})}{(1-p)} \left[\delta(pz/z') - \delta(p^{-1}z/z') \right], \quad (1.1)$$

where $p = q/t$ and

$$f(z) = \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} \frac{(1-q^n)(1-t^{-n})}{(1+p^n)} z^n \right), \quad \delta(z) = \sum_{n \in \mathbb{Z}} z^n. \quad (1.2)$$

¹The root of unity limit of the troidal algebra has also been considered in [45].

It is known that the q -deformed current $\mathcal{T}(z)$ can be realized by the q -deformed Heisenberg algebra. In this paper, we consider the following root of unity limit in q and t :

$$q \rightarrow 1, \quad t \rightarrow -1. \quad (1.3)$$

This limit has been considered in [63] and the q -Virasoro algebra (1.1) is reduced to the Lepowsky-Wilson's Z -algebra [64, 65]. The $\widehat{sl}(2)_k$ current algebra can be realized by using the two types of bosons obtained from a q -deformed boson in this limit and an additional boson. In this formalism, we will first reconstruct the integral representation of the solution to the KZ equation [66] that the current blocks should satisfy. Then we consider the current block with the \mathcal{K} operator insertion and illustrate how to derive those as integral representation.

This paper is organized as follows: In section two, the root of unity limit of the q -deformed boson is considered. In section three, we see that the $\widehat{sl}(2)_k$ current algebra can be realized. In section four, integral formulas of the solutions to the KZ equation are constructed. In section five, we consider the four-point current block with the \mathcal{K} operator insertion and present its integral representation.

2 q -deformed boson at root of unity

As mentioned in Introduction, the q -deformed Virasoro current $\mathcal{T}(z)$ can be realized by the q -deformed Heisenberg algebra [59],

$$\begin{aligned} [\alpha_0, \tilde{Q}] &= 2, \\ [\alpha_n, \alpha_m] &= -\frac{1}{n}(1 - q^{-n})(1 - t^n)(1 + p^n)\delta_{n+m,0}. \end{aligned} \quad (2.1)$$

The q -deformed boson is defined by

$$\tilde{\varphi}(z) = \tilde{\varphi}_{\text{even}}(z) + \tilde{\varphi}_{\text{odd}}(z), \quad (2.2)$$

where

$$\tilde{\varphi}_{\text{even}}(z) = \beta^{\frac{1}{2}}\tilde{Q} + \beta^{\frac{1}{2}}\alpha_0 \log z - \sum_{n \neq 0} \frac{1}{q^n - q^{-n}} \alpha_{2n} z^{-2n}, \quad (2.3)$$

$$\tilde{\varphi}_{\text{odd}}(z) = - \sum_{n \in \mathbf{Z}} \frac{1}{q^{(2n+1)/2} - q^{-(2n+1)/2}} \alpha_{2n+1} z^{-(2n+1)}. \quad (2.4)$$

Let us consider the following limit of the q -deformed boson,

$$q = t^\alpha = e^{-\frac{h}{\sqrt{\beta}}}, \quad t = (-1)^{\tilde{k}} e^{-\sqrt{\beta}h}, \quad p = q/t = (-1)^{-\tilde{k}} e^{Q_E h}, \quad h \rightarrow 0. \quad (2.5)$$

where \tilde{k} is an odd number and $Q_E = \sqrt{\beta} - \frac{1}{\sqrt{\beta}}$ and $\alpha = \frac{1}{\beta}$. In order to take this limit, the following condition is demanded:

$$\alpha \left(i\pi k - \frac{h}{\sqrt{\alpha}} + 2\pi i m_+ \right) = -\sqrt{\alpha} h + 2\pi i m_- \quad (2.6)$$

where m_{\pm} are positive integers. Thus the parameter α has to satisfy

$$\alpha = \frac{1}{\beta} = \frac{2m_-}{2m_+ + \tilde{k}} \equiv \frac{2}{2+k}, \quad (2.7)$$

where

$$k = \frac{2(m_+ - m_-) + \tilde{k}}{m_-}. \quad (2.8)$$

In the section 3, we will see that the parameter k serves as the level of the $\widehat{sl}(2)_k$ algebra. In the limit (2.5), the even and odd part of the q -boson are expanded in powers of h as

$$\tilde{\varphi}_{\text{even}}(z) \equiv \phi_1(z) + \mathcal{O}(h), \quad (2.9)$$

$$\tilde{\varphi}_{\text{odd}}(z) \equiv \phi_2(z) + \mathcal{O}(h), \quad (2.10)$$

where

$$\phi_1(z) = Q + \phi_0^{(1)} \log z - \sum_{n \neq 0} \frac{\phi_{2n}^{(1)}}{2n} z^{-2n}, \quad (2.11)$$

$$\phi_2(z) = - \sum_{n \in \mathbf{Z}} \frac{\phi_{2n+1}^{(2)}}{2n+1} z^{-(2n+1)}. \quad (2.12)$$

with the commutation relations,

$$[\phi_{2n}^{(1)}, \phi_{2m}^{(1)}] = (k+2)(2n)\delta_{n+m,0}, \quad [\phi_0^{(1)}, Q] = k+2, \quad (2.13)$$

$$[\phi_{2n+1}^{(2)}, \phi_{-2m-1}^{(2)}] = -k(2n+1)\delta_{n,m}. \quad (2.14)$$

3 Affine $\widehat{sl}_k(2)$ algebra

3.1 Free field realization

There exists the well-known free field realization in terms of a free boson and β - γ system i.e. the Wakimoto representation [62]. However, we introduce another realization of the $\widehat{sl}_k(2)$ current algebra in terms of the bosons obtained in the previous section because it is useful in order to consider the insertion of the \mathcal{K} operator which we will see in the section 5.

The following additional boson are required:

$$\phi_0(z) = - \sum_{n \in \mathbf{Z}} \frac{\phi_{2n+1}^{(0)}}{2n+1} z^{-(2n+1)}, \quad (3.1)$$

with

$$[\phi_{2n+1}^{(0)}, \phi_{-2m-1}^{(0)}] = k(2n+1)\delta_{n,m}. \quad (3.2)$$

This has the same algebraic structure as $\phi^{(2)}(z)$ (times i). Following [63], let us introduce

$$\beta(z) = \partial\phi_0(z), \quad (3.3)$$

$$x(z) =: (\partial\phi_1(z) + \partial\phi_2(z))e^{\frac{2}{k}\varphi(z)} :, \quad (3.4)$$

where $\varphi(z) = \phi_0(z) + \phi_2(z)$ and the symbol $::$ stands for the normal ordering defined in the standard way. Using $\beta(z)$ and $x(z)$, we can explicitly construct the $\widehat{sl}(2)$ currents of the level k as follows:

$$E(w) = \frac{1}{2}(\beta(z) - x_e(z)) \Big|_{w=z^2}, \quad (3.5)$$

$$F(w) = \frac{1}{2z^2}(\beta(z) + x_e(z)) \Big|_{w=z^2}, \quad (3.6)$$

$$H(w) = \left(\frac{x_o(z)}{z} + \frac{k}{2z^2} \right) \Big|_{w=z^2}, \quad (3.7)$$

where

$$x_e(z) = \frac{1}{2}(x(z) + x(-z)), \quad x_o(z) = \frac{1}{2}(x(z) - x(-z)). \quad (3.8)$$

Note that the currents are defined on w -plane. In fact, one can easily check that $E(w)$, $F(w)$ and $H(w)$ serve precisely as the affine $\widehat{sl}(2)_k$ currents,

$$\begin{aligned} H(w_1)H(w_2) &\sim \frac{2k}{(w_1 - w_2)^2}, \\ H(w_1)E(w_2) &\sim \frac{2}{w_1 - w_2}E(w_2), \\ H(w_1)F(w_2) &\sim \frac{-2}{w_1 - w_2}F(w_2), \\ E(w_1)F(w_2) &\sim \frac{k}{(w_1 - w_2)^2} + \frac{1}{w_1 - w_2}H(w_2). \end{aligned} \quad (3.9)$$

The stress tensor with the central charge $c = \frac{3k}{k+2}$ can also be constructed by the Sugawara construction,

$$\begin{aligned} T(w) = \frac{1}{4z^2} \left\{ \frac{1}{k}(\partial\phi_0)^2 + \frac{1}{\kappa}(\partial\phi_1)^2 - \frac{1}{k}(\partial\phi_2)^2 - \frac{1}{\kappa} \left(\partial + \frac{1}{z} \right) \partial\phi_1 \right. \\ \left. + \frac{1}{z} \left(\partial\phi_1 \cosh \frac{2}{k}\varphi + \partial\phi_2 \sinh \frac{2}{k}\varphi \right) + \frac{k(k+4)}{4\kappa z^2} \right\} \Big|_{w=z^2}, \end{aligned} \quad (3.10)$$

where $\kappa = k + 2$ as usual.

3.2 spin $j/2$ representation

The operators corresponding to the highest (respectively, lowest) weight state of the spin $1/2$ representation are given by

$$V_{1/2,1/2}(w) = \frac{1}{2} : \left(e^{\alpha^i \phi_i(z)} + e^{\alpha^i \phi_i(-z)} \right) : \Big|_{w=z^2} = z^{\frac{k}{2\kappa}} : e^{\frac{1}{\kappa} \phi_1} \cosh \left(\frac{1}{k} \varphi(z) \right) : \Big|_{w=z^2}, \quad (3.11)$$

$$V_{1/2,-1/2}(w) = \frac{1}{2z} : \left(e^{\alpha^i \phi_i(z)} - e^{\alpha^i \phi_i(-z)} \right) : \Big|_{w=z^2} = z^{\frac{k}{2\kappa}} : e^{\frac{1}{\kappa} \phi_1} \frac{\sinh \left(\frac{1}{k} \varphi(z) \right)}{z} : \Big|_{w=z^2}, \quad (3.12)$$

where the repeated indices i are summed over for $i = 0, 1, 2$ and

$$(\alpha^0, \alpha^1, \alpha^2) = \left(\frac{1}{k}, \frac{1}{\kappa}, \frac{1}{k} \right). \quad (3.13)$$

In general, the operators corresponding to the states belonging to the spin $j/2$ representation ($j \in \mathbf{Z}_{\geq 0}$) are given by

$$V_{j/2,j/2-m}(w) = z^{\frac{jk}{2\kappa}-m} : e^{j\alpha^1 \phi_1} \cosh^{j-m} \left(\frac{1}{k} \varphi(z) \right) \sinh^m \left(\frac{1}{k} \varphi(z) \right) : \Big|_{w=z^2}, \quad 0 \leq m \leq j. \quad (3.14)$$

In particular, the operator corresponding to the highest weight state is

$$V_{j/2}(w) \equiv V_{j/2,j/2}(w) = z^{\frac{kj}{2\kappa}} : e^{j\alpha^1 \phi_1} \cosh^j \left(\frac{1}{k} \varphi(z) \right) : \Big|_{w=z^2}. \quad (3.15)$$

The vertex operator $V_{j/2,j/2-m}(w)$ has the expected behavior,

$$\begin{aligned} H(w_1) V_{j/2,j/2-m}(w_2) &\sim \frac{2 \left(\frac{j}{2} - m \right)}{w_1 - w_2} V_{j/2,j/2-m}(w_2), \\ E(w_1) V_{j/2,j/2-m}(w_2) &\sim \frac{m}{w_1 - w_2} V_{j/2,j/2-m+1}(w_2), \\ F(w_1) V_{j/2,j/2-m}(w_2) &\sim \frac{j - m}{w_1 - w_2} V_{j/2,j/2-m-1}(w_2), \end{aligned} \quad (3.16)$$

On the other hand, it is easy to check that $V_{j/2,j/2-m}(w)$ is also primary operator with the scaling dimension $\Delta_j = \frac{j(j+2)}{4\kappa}$.

4 Solutions of KZ equation

Let $V = V_{j_1} \otimes V_{j_2} \otimes \cdots \otimes V_{j_N}$. Here $V_{j_n}, 0 \leq n \leq N$ is the $(j_n + 1)$ -dimensional lowest weight module for \mathfrak{sl}_2 . The standard Chevalley basis of \mathfrak{sl}_2 is denoted by $\{e, f, h\}$ with $[e, f] =$

$h, [h, e] = 2e, [h, f] = -2f$. The lowest weight state $v_j \in V_j$ is defined by

$$f \cdot v_j = 0, \quad h \cdot v_j = -jv_j, \quad e^{j+1} \cdot v_j = 0. \quad (4.1)$$

The KZ equation [66] (for review, see [67]) is written as

$$\kappa \frac{\partial}{\partial w_n} \Psi(\mathbf{w}) = \left(\sum_{m=1, m \neq n}^N \frac{\Omega_{mn}}{w_m - w_n} \right) \Psi(\mathbf{w}), \quad n = 1, \dots, N. \quad (4.2)$$

where $\Psi(\mathbf{w})$ is the function which takes value in V and

$$\Omega_{mn} = e_m f_n + f_m e_n + \frac{1}{2} h_m h_n. \quad (4.3)$$

Here x_n stands for the action of $x \in \mathfrak{sl}_2$ on V_{j_n} , i.e.

$$x_n = 1 \otimes 1 \otimes \dots \otimes \overset{n}{\underset{\vee}{x}} \otimes \dots \otimes 1, \quad (4.4)$$

$$x_m y_n = 1 \otimes 1 \otimes \dots \otimes \overset{m}{\underset{\vee}{x}} \otimes \dots \otimes \overset{n}{\underset{\vee}{y}} \otimes \dots \otimes 1 \quad (4.5)$$

In this section, we want to get the integral formulas of the solutions of (4.2) in the free field realization constructed in the previous section.

4.1 A simple solution

Let us define

$$X(z) =: \cosh \left(\frac{1}{k} (\phi_0 + \phi_2)(z) \right) :, \quad Y(z) = \frac{1}{z} : \sinh \left(\frac{1}{k} (\phi_0 + \phi_2)(z) \right) :, \quad (4.6)$$

$$Z_j(z) = z^{\frac{kj}{2\kappa}} : e^{\frac{j}{\kappa} \phi_1(z)} :. \quad (4.7)$$

Then the operator (3.14) is expressed by

$$V_{j/2, j/2-m}(w) = Z_j(z) X(z)^{j-m} Y(z)^m \Big|_{w=z^2}. \quad (4.8)$$

Note that

$$X(z_1) X(z_2) =: X(z_1) X(z_2) :, \quad Y(z_1) Y(z_2) =: Y(z_1) Y(z_2) :, \quad X(z_1) Y(z_2) =: X(z_1) Y(z_2) :, \quad (4.9)$$

$$\prod_{i=1}^N Z_{j_i}(z_i) = \prod_{m < n}^N (z_m^2 - z_n^2)^{\frac{j_m j_n}{2\kappa}} : \prod_{n=1}^N Z_{j_n}(z_n) :. \quad (4.10)$$

Let us choose the Fock vacuum $|\Omega\rangle$ and the conjugate $\langle\Omega|$ as

$$\begin{aligned}\phi_{2n+1}^{(0)}|\Omega\rangle &= 0, & \phi_{2n}^{(1)}|\Omega\rangle &= 0, & \phi_{2n+1}^{(2)}|\Omega\rangle &= 0, & n \geq 0, \\ \langle\Omega|\phi_{2n+1}^{(0)} &= 0, & \langle\Omega|\phi_{2n}^{(1)} &= 0, & \langle\Omega|\phi_{2n+1}^{(2)} &= 0, & n < 0,\end{aligned}\tag{4.11}$$

The highest weight state of the spin $j/2$ representation is given by

$$|j\rangle = e^{\frac{1}{\kappa}(j-\frac{\kappa}{2})Q}|\Omega\rangle, \quad \langle j| = \langle\Omega|e^{-\frac{1}{\kappa}(j-\frac{\kappa}{2})Q}.\tag{4.12}$$

We denote by \mathcal{H}_j the highest weight module over $\widehat{sl}(2)_\kappa$ generated from $|j\rangle$. Then the operator $V_{j/2, j/2-m}(w)$ plays a role to map \mathcal{H}_{j_0} onto \mathcal{H}_{j_0+j} for any $j_0 \in \mathbf{Z}_{\geq 0}$. From the product of N $V_{j/2}$'s we obtain

$$\langle j| \prod_{n=1}^N V_{j_n/2}(w_n)|0\rangle = \prod_{m < n}^N (w_m - w_n)^{\frac{j_m j_n}{2\kappa}} \equiv \psi_0(\mathbf{w}), \quad j = \sum_{n=1}^N j_n.\tag{4.13}$$

Consequently the simple solution of (4.2) is given by

$$\Psi_0(\mathbf{w}) = \psi_0(\mathbf{w})v = \prod_{m < n}^N (w_m - w_n)^{\frac{j_m j_n}{2\kappa}} v = \langle j| \prod_{n=1}^N V_{j_n/2}(z_n)|0\rangle v, \quad j = \sum_{n=1}^N j_n,\tag{4.14}$$

where $v = v_{j_1} \otimes v_{j_2} \otimes \cdots \otimes v_{j_N} \in V$.

4.2 screening charge

The screening current is defined by ²

$$S(\tau) = t^{-\frac{\kappa}{2}} \partial \phi_2(t) e^{-\frac{2}{\kappa} \phi_1(t)} \Big|_{\tau=t^2},\tag{4.15}$$

which satisfies

$$\begin{aligned}\beta(z)S(\tau) &\sim 0, \\ x_e(z)S(\tau) &\sim \frac{\partial}{\partial \tau} A_e(z, \tau), \\ \frac{x_o(z)}{z}S(\tau) &\sim \frac{\partial}{\partial \tau} A_o(z, \tau),\end{aligned}\tag{4.16}$$

²The screening current (4.15) can be obtained by taking the root of unity limit of the q -deformed screening current $\tilde{S}(z) = e^{\tilde{\varphi}(z)}$ up to the overall factor.

where

$$A_e(z, \tau) = \kappa \frac{t^{\frac{k+4}{k+2}}}{z^2 - t^2} \left(e^{\frac{2}{k}(\phi_0 + \phi_2)(t)} + e^{-\frac{2}{k}(\phi_0 + \phi_2)(t)} \right) e^{-\frac{2}{k+2}\phi_1(t)} \Big|_{\tau=t^2}, \quad (4.17)$$

$$A_o(z, \tau) = \kappa \frac{t^{\frac{k+4}{k+2}}}{z^2 - t^2} \frac{1}{t} \left(e^{\frac{2}{k}(\phi_0 + \phi_2)(t)} - e^{-\frac{2}{k}(\phi_0 + \phi_2)(t)} \right) e^{-\frac{2}{k+2}\phi_1(t)} \Big|_{\tau=t^2}. \quad (4.18)$$

The screening charge defined by

$$U = \int_C d\tau S(\tau), \quad (4.19)$$

commutes with the $\widehat{sl}(2)_k$ currents $E(w), H(w), F(w)$. Here we postulate the cycle C on w -plane is chosen appropriately.

4.3 intertwining operator

In this section, we consider the intertwining operator $\Phi_j^m(z) : \mathcal{H}_{j_0} \rightarrow \mathcal{H}_{j_0+j-2m} \otimes V_j$, $\forall j_0 \in \mathbf{Z}_{\geq 0}$. Let us introduce the following formal operator:

$$\gamma(z) = X(z)^{-1} Y(z). \quad (4.20)$$

The intertwining operator of level $m = 0$ can be constructed in terms of $Z_j(z)$, $X(z)$ and $Y(z)$ as

$$\Phi_j^0(z)u = Z_j(z)X^j(z)e^{-\gamma(z) \otimes e}(u \otimes v_j), \quad (4.21)$$

where $u \in \mathcal{H}_{j_0}$ and $v_j \in V_j$ is the lowest weight state. In fact, $\Phi_j^0(z)$ satisfies the intertwining relation,

$$\Phi_j^0(z)J_n^A = (J_n^A \otimes 1 + (z^2)^n \cdot 1 \otimes A)\Phi_j^0(z) \quad A = e, f, h. \quad (4.22)$$

By using the intertwining operators, $\Psi_0(\mathbf{w})$ is given by

$$\Psi_0(\mathbf{w}) = \langle j | \prod_{n=1}^N \Phi_{j_n}^0(z_n) | 0 \rangle, \quad j = \sum_{n=1}^N j_n. \quad (4.23)$$

The general m intertwining operator can be obtained by multiplying m screening charges to $\Phi_j^0(z)$,

$$\Phi_j^m(z)u = \Phi_j^0(z)U^m, \quad (4.24)$$

and the solutions of the KZ equation are given by

$$\Psi_m(\mathbf{w}) = \langle j - 2m | \prod_{i=1}^N \Phi_{j_i}^{m_i}(z_i) | 0 \rangle, \quad j = \sum_{i=1}^N j_i, \quad m = \sum_{i=1}^N m_i. \quad (4.25)$$

which reproduces the well-known integral formulas. For example, we obtain, in the case of $m = 1$,

$$\begin{aligned}\Psi_1(\mathbf{w}) &= \langle j-2 | \prod_{n=1}^{N-1} \Phi_{j_n}^0(z_n) \Phi_{j_N}^1(z_N) | 0 \rangle \\ &= \prod_{m < n}^N (w_m - w_n)^{\frac{imjn}{2\kappa}} \int_C d\tau \prod_{n=1}^N (w_n - \tau)^{-\frac{in}{\kappa}} \left(\sum_{n=1}^N \frac{-e_n}{w_n - \tau} \right) v,\end{aligned}\quad (4.26)$$

and in the case of $m = 2$,

$$\begin{aligned}\Psi_2(\mathbf{w}) &= \langle j-4 | \prod_{n=1}^{N-1} \Phi_{j_n}^0(z_n) \Phi_{j_N}^2(z_N) | 0 \rangle \\ &= \prod_{m < n}^N (w_m - w_n)^{\frac{imjn}{2\kappa}} \int_C d\tau \prod_{n=1}^N \prod_{p=1}^2 (w_n - \tau_p)^{-\frac{in}{\kappa}} (\tau_1 - \tau_2)^{\frac{2}{\kappa}} \\ &\quad \times \left(\sum_{m=1}^N \sum_{n=1}^N \frac{e_m e_n}{(w_m - \tau_1)(w_n - \tau_2)} \right) v,\end{aligned}\quad (4.27)$$

where $\boldsymbol{\tau} = (\tau_1, \tau_2)$ and $C = C_1 \times C_2$. Here we have used

$$\Phi_j^0(z) S(\tau) = (w - \tau)^{-\frac{j}{\kappa}} \left\{ : \Phi_j^0(z) S(\tau) : - \frac{\tau^{-\frac{k}{2\kappa}}}{w - \tau} ((1 \otimes e) - (w \otimes f)) : e^{-\frac{2}{\kappa} \phi_1(t)} \Phi_j^0(z) : \right\}, \quad (4.28)$$

$$S(\tau) S(\tau') = (\tau - \tau')^{\frac{2}{\kappa}} \left\{ : S(\tau) S(\tau') : - \frac{k(\tau\tau')^{-\frac{k}{2\kappa}}(\tau + \tau')}{(\tau - \tau')^2} : e^{-\frac{2}{\kappa}(\phi_1(t) + \phi_1(t'))} : \right\}. \quad (4.29)$$

5 Insertion of \mathcal{K} operator

In the paper [55], the authors proposed the insertion of the \mathcal{K} operator to the $\widehat{sl}(2)_k$ current block current block in order to establish the AGT relation in the presence of a full surface operator. The generalization to the $\widehat{sl}(n)_k$ current block is also discussed in [56]. The \mathcal{K} operator is defined by ³

$$\mathcal{K}^\dagger = \exp \left(\sum_{n=1}^{\infty} \frac{1}{2n-1} [J_{n-1}^+ + J_n^-] \right). \quad (5.1)$$

In our formalism, the \mathcal{K} operator has the following simple expression in terms of $\phi_0(z)$:

$$\mathcal{K}^\dagger = e^{-\phi_0^{(+)}(1)}, \quad (5.2)$$

³Since the \mathcal{K} operator we will use below is equal to $\mathcal{K}^\dagger(1, 1)$ presented in [56], we denote it by \mathcal{K}^\dagger .

where $\phi_0^{(+)}(z)$ is the positive modes of $\phi_0(z)$.⁴ Therefore it is meaningful to consider the four-point correlation function with a \mathcal{K} operator insertion,

$$\tilde{\Psi}_{m_1, m_2}(w_2) \equiv \langle j - 2m | \Phi_{j_1}^{m_1}(1) e^{-\phi_0^{(+)}(1)} \Phi_{j_2}^{m_2}(z_2) | j_3 \rangle. \quad (5.3)$$

Since the screening current $S(z)$ does not include $\phi_0(z)$, it is trivial that the \mathcal{K} operator commutes with U . However, the action of $e^{-\phi_0^{(+)}(1)}$ on $\Phi_j^0(z)$ yields nontrivial result and we obtain

$$\tilde{\Psi}_{m_1, m_2}(w_2) = \langle j - 2m | \Phi_{j_1}^{m_1}(1) \tilde{\Phi}_{j_2}^{m_2}(z_2) | j_3 \rangle, \quad (5.4)$$

where

$$\begin{aligned} \tilde{\Phi}_j^m(z)u &= \tilde{\Phi}_j^0(z)U^m u \\ &= Z_j(z)\tilde{X}(z)e^{-\tilde{\gamma}(z)\otimes e}U^m u \otimes v_j, \end{aligned} \quad (5.5)$$

with

$$\begin{aligned} \tilde{X}(z) &= X(z) + z^2 Y(z), \\ \tilde{Y}(z) &= Y(z) + X(z), \\ \tilde{\gamma}(z) &= \tilde{X}^{-1}(z)\tilde{Y}(z). \end{aligned} \quad (5.6)$$

It is easy to get the explicit integral formulas for the four point current block with the \mathcal{K} operator insertion. For example,

$$\tilde{\Psi}_{0,0}(w_2) = (1 - w_2)^{\frac{j_1 j_2}{2\kappa} - \frac{j_2}{2}} w_2^{\frac{j_2 j_3}{2\kappa}} (v_{j_1} \otimes e^{-e} v_{j_2}), \quad (5.7)$$

$$\begin{aligned} \tilde{\Psi}_{0,1}(w_2) &= (1 - w_2)^{\frac{j_1 j_2}{2\kappa} - \frac{j_2}{2}} w^{\frac{j_2 j_3}{2\kappa}} \int_C d\tau (1 - \tau)^{-\frac{j_1}{\kappa}} (w_2 - \tau)^{-\frac{j_2}{\kappa}} \tau^{-\frac{j_3}{\kappa}} \\ &\quad \left\{ \frac{-e_1}{1 - \tau} + \frac{-e_2 + w_2 f_2}{w_2 - \tau} \right\} (v_{j_1} \otimes e^{-e} v_{j_2}). \end{aligned} \quad (5.8)$$

6 Summary

To summarize, we have reproduced the free field realization of the $sl(2)_k$ current algebra and the integral formulas of the KZ equation by using three types of chiral bosons which are obtained from the q -deformed boson in the root of unity limit. In addition, we have derived the integral formulas for the modified four point current blocks.

⁴The original \mathcal{K} operator in [55] can be realized by the negative modes of ϕ_0 .

Finally, we give a comment on the modification of KZ equation. The insertion of the \mathcal{K} operator would modify the original KZ equation. Now, let us examine the OPE with the stress-energy tensor (3.10),

$$T(w_1)\tilde{\Phi}_j^0(w_2) \sim \frac{1}{(w_1 - w_2)^2} \frac{j(j+2)}{4\kappa} \tilde{\Phi}_j^0(w_2) + \frac{1}{w_1 - w_2} \partial_{w_2} \tilde{\Phi}_j^0(w_2) - \frac{1}{w_1 - w_2} \left(j\tilde{X}^{-1}Y \otimes 1 + \tilde{X}^{-2}\tilde{Y}Y \otimes e \right) \tilde{\Phi}_j^0(w_2). \quad (6.1)$$

The last term is the additional one. The OPEs with $H(w)$, $F(w)$ and $E(w)$ have also the extra terms. The modification of the KZ equation is provided by these unusual behaviors in $\tilde{\Phi}_j^0(z)$ with respect to the Virasoro algebra and the current algebra. Such an equation may have the solution displayed in this paper, for example, (5.7) and (5.8) and may be related to the quantum isomonodromy equation proposed in [68].

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References

- [1] L. F. Alday, D. Gaiotto and Y. Tachikawa, “Liouville Correlation Functions from Four-dimensional Gauge Theories,” *Lett. Math. Phys.* **9**, 167-197 (2010) [arXiv:0906.3219 [hep-th]].
- [2] N. Nekrasov, “Seiberg-Witten prepotential from instanton counting,” *Adv. Theor. Math. Phys.* **7**, 831-864 (2004) [arXiv:hep-th/0206161].
- [3] A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov, “Infinite conformal symmetry in two-dimensional quantum field theory,” *Nucl. Phys. B* **241**, 333-380 (1984).
- [4] N. Wyllard, “ A_{N-1} conformal Toda field theory correlation functions from conformal $\mathcal{N} = 2$ $SU(N)$ quiver gauge theories,” *JHEP* **0911**, 002 (2009) [arXiv:0907.2189 [hep-th]].
- [5] A. Mironov and A. Morozov, “On AGT relation in the case of $U(3)$,” *Nucl. Phys. B* **825**, 1-37 (2010) [arXiv:0908.2569 [hep-th]].
- [6] L. Hollands, C. A. Keller, J. Song, “From SO/Sp instantons to W -algebra blocks,” *JHEP* **1103** (2011) 053 [arXiv:1012.4468 [hep-th]].

- [7] L. Hollands, C. A. Keller, J. Song, “Towards a 4d/2d correspondence for Sicilian quivers,” JHEP 1110 (2011) 100 [arXiv:1107.0973 [hep-th]].
- [8] The ABCDEFG of Instantons and W-algebras C. A. Keller, N. Mekareeya, J. Song and Y. Tachikawa, JHEP 1203 (2012) 045 [arXiv:1111.5624 [hep-th]].
- [9] R. Dijkgraaf and C. Vafa, “Toda Theories, Matrix Models, Topological Strings, and $N = 2$ Gauge Systems,” [arXiv:0909.2453 [hep-th]].
- [10] H. Itoyama, K. Maruyoshi and T. Oota, “The Quiver Matrix Model and 2d-4d Conformal Connection,” Prog. Theor. Phys. **123**, 957-987 (2010) [arXiv:0911.4244 [hep-th]].
- [11] A. Mironov, A. Morozov and Sh. Shakirov, “Matrix Model Conjecture for Exact BS Periods and Nekrasov Functions,” JHEP **1002**, 030 (2010) [arXiv:0911.5721 [hep-th]].
- [12] V. A. Fateev and A. V. Litvinov, “On AGT conjecture,” JHEP 1002 (2010) 014 [arXiv:0912.0504 [hep-th]].
- [13] A. Mironov, A. Morozov and Sh. Shakirov, “Conformal blocks as Dotsenko-Fateev Integral Discriminants,” J. Mod. Phys. A **25**, 3173-3207 (2010) [arXiv:1001.0563 [hep-th]].
- [14] H. Itoyama and T. Oota, “Method of generating q -expansion coefficients for conformal block and $\mathcal{N} = 2$ Nekrasov function by β -deformed matrix model,” Nucl. Phys. B **838**, 298-330 (2010) [arXiv:1003.2929 [hep-th]].
- [15] A. Mironov, A. Morozov and And. Morozov, “Matrix model version of AGT conjecture and generalized Selberg integrals,” Nucl. Phys. B **843**, 534-557 (2011) [arXiv:1003.5752 [hep-th]].
- [16] H. Itoyama, T. Oota and N. Yonezawa, “Massive scaling limit of the β -deformed matrix model of Selberg type,” Phys. Rev. D **82**, 085031 (2010) [arXiv:1008.1861 [hep-th]].
- [17] A. Mironov, A. Morozov and Sh. Shakirov, “A direct proof of AGT conjecture at $\beta = 1$,” JHEP **1102**, 067 (2011) [arXiv:1012.3137 [hep-th]].
- [18] G. Bonelli, K. Maruyoshi and A. Tanzini, “Quantum Hitchin Systems via beta-deformed Matrix Models,” arXiv:1104.4016 [hep-th].
- [19] S. Kanno, Y. Matsuo and H. Zhang, “Extended Conformal Symmetry and Recursion Formulae for Nekrasov Partition Function,” JHEP **1308**, 028 (2013) [arXiv:1306.1523 [hep-th]].
- [20] A. Morozov and A. Smirnov, “Towards the Proof of AGT Relations with the Help of the Generalized Jack Polynomials,” Lett. Math. Phys. **104** 585-612 (2014) [arXiv:1307.2576 [hep-th]].
- [21] S. Mironov, And. Morozov and Y. Zenkevich, “Generalized Jack polynomials and the AGT relations for the SU(3) group,” JETP Lett. **99**, 109-113 (2014) [arXiv:1312.5732 [hep-th]].
- [22] Y. Matsuo, C. Rim and H. Zhang, “Construction of Gaiotto states with fundamental multiplets through Degenerate DAHA,” JHEP **1409** (2014) 028 [arXiv:1405.3141 [hep-th]].
- [23] V. Belavin and B. Feigin, “Super Liouville conformal blocks from $N = 2$ SU(2) quiver gauge theories,” JHEP **1107**, 079 (2011) [arXiv:1105.5800 [hep-th]].

- [24] T. Nishioka and Y. Tachikawa, “Central charges of para-Liouville and Toda theories from M-5-branes,” *Phys. Rev. D* **84**, 046009 (2011) [arXiv:1106.1172 [hep-th]].
- [25] G. Bonelli, K. Maruyoshi and A. Tanzini, “Instantons on ALE spaces and Super Liouville Conformal Field Theories,” *JHEP* **1108** (2011) 056 [arXiv:1106.2505 [hep-th]].
- [26] G. Bonelli, K. Maruyoshi and A. Tanzini, “Gauge Theories on ALE Space and Super Liouville Correlation Functions,” *Lett. Math. Phys.* **101** (2012) 103-124 [arXiv:1107.4609 [hep-th]].
- [27] A. Belavin, V. Belavin and M. Bershtein, “Instantons and 2d Superconformal field theory,” *JHEP* **1109**, 117 (2011) [arXiv:1106.4001 [hep-th]].
- [28] N. Wyllard, “Coset conformal blocks and $\mathcal{N} = 2$ gauge theories,” arXiv:1109.4264 [hep-th].
- [29] Y. Ito, “Ramond sector of super Liouville theory from instantons on an ALE space,” *Nucl. Phys. B* **861** (2012) 387-402 [arXiv:1110.2176 [hep-th]].
- [30] M. N. Alfimov and G. M. Tarnopolsky, “Parafermionic Liouville field theory and instantons on ALE spaces,” *JHEP* **1202** (2012) 036 [arXiv:1110.5628 [hep-th]].
- [31] V. Belavin and N. Wyllard, “ $\mathcal{N} = 2$ superconformal blocks and instanton partition functions,” *JHEP* **1206** (2012) 173 [arXiv:1205.3091 [hep-th]].
- [32] A. Belavin and B. Mukhametzhanov, “ $\mathcal{N} = 1$ superconformal blocks with Ramond fields from AGT correspondence,” *JHEP* **1301** (2013) 178 [arXiv:1210.7454 [hep-th]].
- [33] H. Awata and Y. Yamada, “Five-dimensional AGT Conjecture and the Deformed Virasoro Algebra,” *JHEP* **1001** (2010) 125 [arXiv:0910.4431 [hep-th]].
- [34] H. Awata and Y. Yamada, “Five-Dimensional AGT Relation and the Deformed β -Ensemble,” *Prog. Theor. Phys.* **124**, 227-262 (2010) [arXiv:1004.5122 [hep-th]].
- [35] H. Awata, B. Feigin, A. Hoshino, M. Kanai, J. Shiraishi and S. Yanagida, “Notes on Ding-Iohara algebra and AGT conjecture,” arXiv:1106.4088 [math-ph].
- [36] A. Mironov, A. Morozov, S. Shakirov, and A. Smirnov, “Proving AGT conjecture as HS duality: extension to five dimensions,” *Nucl. Phys. B* **855** (2012) 128-151 [arXiv:1105.0948 [hep-th]].
- [37] Y. Ohkubo, “Existence and Orthogonality of Generalized Jack Polynomials and Its q -Deformation,” arXiv:1404.5401 [math-ph].
- [38] Y. Zenkevich, “Generalized Macdonald polynomials, spectral duality for conformal blocks and AGT correspondence in five dimensions,” *JHEP* **1405** (2015) 131 [arXiv:1412.8592 [hep-th]].
- [39] A. Morozov and Y. Zenkevich, “Decomposing Nekrasov Decomposition,” arXiv:1510.01896 [hep-th].
- [40] A. Iqbal, C. Kozçaz and S.-T. Yau, “Elliptic Virasoro Conformal Blocks,” arXiv:1511.00458 [hep-th].
- [41] F. Nieri, “An elliptic Virasoro symmetry in 6d,” arXiv:1511.00574 [hep-th].

- [42] H. Itoyama, T. Oota and R. Yoshioka, “2d-4d Connection between q -Virasoro/W Block at Root of Unity Limit and Instanton Partition Function on ALE Space,” Nucl. Phys. B **877**, 506-537 (2013) [arXiv:1308.2068 [hep-th]].
- [43] H. Itoyama, T. Oota and R. Yoshioka, “ q -Virasoro/W Algebra at Root of Unity and Parafermions,” Nucl. Phys. B **889** 25-35 (2014) [arXiv:1408.4216 [hep-th]].
- [44] H. Itoyama, T. Oota, R. Yoshioka, “ q -Virasoro algebra at root of unity limit and 2d-4d connection,” J. Phys. Conf. Ser. **474** (2013) 012022.
- [45] A. A. Belavin, M. A. Bershtein and G. M. Tarnopolsky, “Bases in coset conformal field theory from AGT correspondence and Macdonald polynomials at the roots of unity,” JHEP 1303 (2013) 019 [arXiv:1211.2788 [hep-th]].
- [46] L. F. Alday, D. Gaiotto, S. Gukov, Y. Tachikawa and H. Verlinde, “Loop and surface operators in $\mathcal{N} = 2$ gauge theory and Liouville modular geometry,” JHEP **1001**, 113 (2010) [arXiv:0909.0945 [hep-th]].
- [47] S. Gukov, “Surface Operators,” Math. Phys. Stud. (2016) 223-259 [arXiv:1412.7127[hep-th]].
- [48] C. Kozçaz, S. Pasquetti and N. Wyllard, “A & B model approaches to surface operators and Toda theories,” JHEP **1008** (2010) 042 [arXiv:1004.2025 [hep-th]].
- [49] T. Dimofte, S. Gukov and L. Hollands, “Vortex Counting and Lagrangian 3-manifolds,” Lett. Math. Phys. **98** (2011) 225-287 [arXiv:1006.0977 [hep-th]].
- [50] K. Maruyoshi and M. Taki, “Deformed Prepotential, Quantum Integrable System and Liouville Field Theory,” Nucl.Phys. B **841** (2010) 388-425 [arXiv:1006.4505 [hep-th]].
- [51] M. Taki, “Surface Operator, Bubbling Calabi-Yau and AGT Relation,” JHEP 1107 (2011) 047 [arXiv:1007.2524 [hep-th]].
- [52] D. Gaiotto, “Asymptotically free $\mathcal{N} = 2$ theories and irregular conformal blocks,” J. Phys. Conf. Ser. **462** (2013) 1, 012014 [arXiv:0908.0307 [hep-th]].
- [53] A. Marshakov, A. Mironov and A. Morozov, “On non-conformal limit of the AGT relations,” Phys. Lett. B **682**, 125 (2009) [arXiv:0909.2052 [hep-th]].
- [54] H. Awata, H. Fuji, H. Kanno, M. Manabe, Y. Yamada, “Localization with a Surface Operator, Irregular Conformal Blocks and Open Topological String,” Adv. Theor. Math. Phys. **16** (2012) 3, 725-804 [arXiv:1008.0574 [hep-th]].
- [55] L. F. Alday and Y. Tachikawa, “Affine $SL(2)$ conformal blocks from 4d gauge theories,” Lett.Math.Phys. **94** (2010) 87-114 [arXiv:1005.4469 [hep-th]].
- [56] C. Kozçaz, S. Pasquetti, F. Passerini and N. Wyllard, “Affine $sl(N)$ conformal blocks from $\mathcal{N} = 2$ $SU(N)$ gauge theories” JHEP 1101 (2011) 045 [arXiv:1008.1412 [hep-th]].
- [57] N. Wyllard, “W-algebras and surface operators in $\mathcal{N} = 2$ gauge theories,” J.Phys. A44 (2011) 155401 [arXiv:1011.0289 [hep-th]].

- [58] N. Wyllard, “Instanton partition functions in $\mathcal{N} = 2$ SU(N) gauge theories with a general surface operator, and their W-algebra duals,” JHEP 1102 (2011) 114 [arXiv:1012.1355 [hep-th]].
- [59] J. Shiraishi, H. Kubo, H. Awata and S. Odake, “A quantum deformation of the Virasoro algebra and the Macdonald symmetric functions,” Lett. Math. Phys. **38**, 33-51 (1996) [arXiv:q-alg/9507034].
- [60] B. Feigin and E. Frenkel, “Quantum \mathcal{W} -Algebras and Elliptic Algebras,” Commun. Math. Phys. **178**, 653-678 (1996) [arXiv:q-alg/9508009].
- [61] H. Awata, H. Kubo, S. Odake and J. Shiraishi, “Quantum \mathcal{W}_N Algebras and Macdonald Polynomials,” Commun. Math. Phys. **179**, 401-416 (1996) [arXiv:q-alg/9508011].
- [62] M. Wakimoto, “Fock representations of the affine Lie algebra $A_1^{(1)}$,” Comm. Math. Phys. 104, (1986), 605-609.
- [63] Y. Hara, M. Jimbo, H. Konno, S. Odake and J. Shiraishi, “On Lepowsky-Wilson’s Z-algebra,” Recent Developments in Infinite-Dimensional Lie Algebras and Conformal Field Theory, Proceedings of the Conference on Infinite-dimensional Lie Theory, Contemporary Mathematics 297, AMS, Providence, 2002, [arXiv:math/0005203].
- [64] J. Lepowsky and R. L. Wilson, “Construction of the affine Lie algebra $A_1^{(1)}$,” Commun. Math. Phys. **62** (1978) 43-53.
- [65] J. Lepowsky and R. L. Wilson, “A new family of algebras underlying the Rogers-Ramanujan identities and generalizations,” Proc. Natl. Acad. Sci. USA, **78** (1981) 7254-7258.
- [66] V.G. Knizhnik and A.B. Zamolodchikov, “Current algebra and Wess-Zumino model in two dimensions,” Nucl. Phys. B **247**, (1984) 83-103.
- [67] P. I. Etingof, I. B. Frenkel and A. A. Kirillov, “*Lectures on Representation Theory and Knizhnik-Zamolodchikov Equations*,” American Mathematical Society, 1997.
- [68] Y. Yamada, “A quantum isomonodromy equation and its application to $\mathcal{N} = 2$ SU(N) gauge theories,” J. Phys. A **44** (2011) 055403 [arXiv:1011.0292 [hep-th]].